

A CRITERIUM OF SINGULARITY FORMATION FOR THE NON-ISENTROPIC GAS DYNAMICS EQUATIONS

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ABSTRACT. For the 1D non-isentropic polytropic gas dynamics equations we find sufficient and necessary conditions for blow up of derivatives in the terms of Cauchy data. In particular, the method allows to determine exact class of initial data corresponding to globally smooth in time solution.

1. PRELIMINARIES

We consider the system of non-isentropic gas dynamics equations for unknown functions ρ, v, S, p (density, velocity, entropy and pressure), namely

$$(1) \quad \rho(\partial_t v + v \partial_x v) + \partial_x p = 0,$$

$$(2) \quad \partial_t \rho + \partial_x(\rho v) = 0,$$

$$(3) \quad \partial_t S + v \partial_x S = 0.$$

The functions depend on time $t \geq 0$ and on point $x \in \mathbb{R}$.

The state equation is

$$(4) \quad p = \frac{1}{\gamma} \rho^\gamma e^S,$$

where $\gamma > 1$ is the adiabatic exponent.

For classical solutions equations (2) – (4) imply

$$(5) \quad \partial_t p + v \partial_x p + \gamma p \partial_x v = 0.$$

We consider the Cauchy problem for (1), (2), (5) with the data

$$(6) \quad \rho(0, x) = \rho_0(x) > 0, \quad v(0, x) = v_0(x), \quad p(0, x) = p_0(x) > 0.$$

System (1) – (3) is symmetric hyperbolic, therefore, it has a local in t solution as smooth as initial data [9]. If we require the initial data (6) to be of class $C^k(\mathbb{R}^n)$, $k > 1$, then the solution keeps this smoothness till it is bounded together with its first derivatives [14],[8]. Further we set $k = 1$.

It is well known that the solutions of the systems of gas dynamics equations, being arbitrary smooth initially, can generate unboundedness of first derivatives within a finite time. This phenomenon is called the gradient catastrophe.

However, the sufficient and necessary conditions for the gradient catastrophe in terms of initial data were known only for the isentropic one-dimensional flow, where one can write the system of two quasilinear equations in Riemannian invariants. For the non-isentropic case quite numerous but only partial results were obtained. The nonlinear capacity method gives sufficient conditions for the loss of smoothness [16]. There are attempts to adapt the characteristics method, but the criterium of the gradient catastrophe was not attained on the way [13], [17], [19].

Recently some progress in the study the non-isentropic gas dynamics equation has been made. Namely, based on the Lagrangian form of the full Euler equations in [4] the authors reduced non-isentropic gas dynamics equations to a special form that allows to study the system by analogy with the Riemann invariants under additional assumptions. The method gives a possibility to find conditions for the singularity formation and to present several new examples of shock-free solutions, which demonstrate a large variety of behaviors [3], [5], [6], [18]. However, the complete picture of the finite time shock formation from smooth initial data was not achieved.

The result of this paper is obtained by a very classical method. We consider the augmented system that consists of the equations for the first derivatives of solution together with the initial system of gas dynamics. This system was introduced in [7] and was used there to prove a local smooth solvability of the Cauchy problem. Further, in [15] it was noticed that this system can always be written in Riemann invariants. To obtain this system of Riemann invariants for the specific case of the gas dynamics equations one can perform standard but thorough computations. For the case of gas dynamics this system can be in some sense decoupled. This is the key point that allows us to reduce the problem to studying an autonomous system of three ordinary differential equations that can be integrated.

The paper is organized as follows. In Sec.2 we formulate the main results. In Sec.3 we recall the criterium of the singularity formation for the isentropic gas dynamics and show that this criterium is a particular case of our main theorem. In Sec.4 we prove the theorem in the general case.

2. MAIN THEOREM

Let us denote $R_1(x) = v'_0$, $R_2(x) = \frac{p'_0}{\sqrt{\gamma\rho_0 p_0}}$, $b(x) = -\frac{\gamma-1}{2} + K(x)$,
 $K(x) = \frac{\gamma p_0 \rho'_0}{2\rho_0 p'_0} - \frac{1}{2}$.

Theorem 1. *Suppose initial data (6) to be of class $C^1(\mathbb{R})$. The solution to (1), (2), (5) keeps smoothness for all $t \geq 0$ if and only if for every point $x \in \mathbb{R}$ one of the following set of inequalities holds:*

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- (7) $b \geq 0, \quad R_1 \geq 0,$
-
- (8) $b > 0, \quad R_2 \neq 0,$
-
- (9) $b < 0, \quad R_1 \geq 0, \quad R_1^2 + \frac{2b}{\gamma-1} R_2^2 \geq 0,$
-
- (10) $R_2 = 0, \quad R_1 \geq 0.$

For all others initial data the derivatives of solution become unbounded within a finite time $T > 0$.

Away from vacuum, the development of singularity in the smooth solution to the compressible Euler system is due to unboundedness of derivatives [11].

Let us remark that the density and pressure taking part of the smooth solution cannot vanish before the singularity formation. Indeed, if the derivatives of smooth solution are bounded, then (2), (5), (6) imply $\rho(t, x) > \phi(t) > 0$, $p(t, x) > \psi(t) > 0$.

If the derivatives of solutions are bounded for all $t > 0$, then the solution keeps smoothness globally in t .

3. ISENTROPIC CASE

In the isentropic case $S = \text{const}$ the system (1), (2), (3) consists of two equations and therefore it can be written in Riemann invariants.

The criterium of gradient catastrophe for the system of two Riemann invariants is known since [2] (see also [15]). Let us recall this theorem and the proof.

Theorem 2. [2] *Consider the system*

$$(11) \quad \partial_t r_k + \xi_k \partial_x r_k = 0, \quad k = 1, 2,$$

subject to the Cauchy data

$$r_k(x, 0) = r_k^0(x) \in C^1(\mathbb{R}), \quad k = 1, 2.$$

Let $r_k = r_k(t, x)$, $\xi_k = \xi_k(r_1, r_2)$ be differentiable functions and

$$\frac{\partial \xi_k}{\partial r_k} > 0, \quad k = 1, 2.$$

If

$$\min_x \min_k \frac{dr_k^0(x)}{dx} < 0,$$

then the derivatives of the solution become unbounded for some $t = T > 0$. Otherwise, if

$$\min_x \min_k \frac{dr_k^0(x)}{dx} \geq 0,$$

then the solution keeps smoothness for all $t > 0$.

Let us recall the idea of proof [2] (see also [15], Ch.1, Sec.10.2). Differentiating (11) with respect to x we get

$$(12) \quad \partial_t p_1 + \xi_1 \partial_x p_1 = -\frac{\partial \xi_1}{\partial r_1} p_1^2 - \frac{\partial \xi_1}{\partial r_2} p_1 p_2,$$

$$(13) \quad \partial_t p_2 + \xi_2 \partial_x p_2 = -\frac{\partial \xi_2}{\partial r_2} p_2^2 - \frac{\partial \xi_2}{\partial r_1} p_1 p_2,$$

where $p_k = \partial_x r_k$. Further, introducing positive functions

$$\phi_1(r_1, r_2) = \exp \left(\int_0^{r_2} \frac{\partial \xi_1}{\partial \tilde{r}_2} \frac{1}{\xi_1 - \xi_2} d\tilde{r}_2 \right),$$

$$\phi_2(r_1, r_2) = \exp \left(\int_0^{r_1} \frac{\partial \xi_2}{\partial \tilde{r}_1} \frac{1}{\xi_2 - \xi_1} d\tilde{r}_1 \right),$$

we obtain from (12), (13)

$$\partial_t(\phi_k p_k) + \xi_k \partial_x(\phi_k p_k) = -\chi_k(\phi_k p_k)^2, \quad \chi_k = \frac{\partial \xi_k}{\partial r_k} / \phi_k, \quad k = 1, 2.$$

The conclusion of Theorem 2 follows immediately. \square

The Riemann invariants for the isentropic gas dynamics are

$$r_{1,2} = v \mp \int_0^p \frac{c(\tilde{\rho})}{\tilde{\rho}} d\tilde{\rho},$$

where $c(\rho) = \sqrt{p_\rho}$ [15]. Computations show that if we set $S = 0$, then

$$r_{1,2} = v \mp \frac{2}{\gamma - 1} \rho^{\frac{\gamma-1}{2}},$$

and the system in the variables r_1, r_2 has the form

$$(14) \quad \partial_t r_1 + (\alpha r_1 + \beta r_2) \partial_x r_1 = 0, \quad \partial_t r_2 + (\beta r_1 + \alpha r_2) \partial_x r_2 = 0,$$

with $\alpha = \frac{1}{2} + \frac{\gamma-1}{4} > 0$, $\beta = \frac{1}{2} - \frac{\gamma-1}{4}$. Thus, in the case of (14) Theorem 2 implies the following corollary.

Corollary 1. *The Cauchy problem (6) in the isentropic case $S = \text{const}$ has a globally smooth solution if and only if*

$$(15) \quad \min_x (v'_0 \mp \rho_0^{\frac{\gamma-3}{2}} \rho'_0) \geq 0$$

Otherwise, the derivatives of solution go to infinity within a finite time $T > 0$.

It can be readily shown that this result follows from Theorem 1 as well. Indeed, if $p = \frac{1}{\gamma} \rho^\gamma$, then $K(x) = 0$, $b(x) < 0$. Condition (9) implies

$$\min_x ((v'_0)^2 - \rho^{\gamma-3} (\rho'_0)^2) \geq 0 \quad \text{and} \quad \min_x v'_0 > 0,$$

it can be reduced to (15).

Remark 1. *The criterium of the singularity formation for the 1D isentropic gas dynamics can be obtained by means of different tools, see e.g. [5].*

4. GENERAL CASE

In the non-isentropic case the system of gas dynamics cannot be written in Riemann invariants. Nevertheless, the augmented system that includes the components of solutions together with their first derivatives can always be written in the Riemann invariants [15]. This fact was known, but rarely used, since the resulting system is nonhomogeneous and it seems there is no hope to analyze it. We are going to show that for the case of the gas dynamics this system can be written in a reasonable form.

4.1. Augmented system and its Riemann invariants. Let us recall the method of obtaining the system of Riemann invariants for any hyperbolic system [15], Ch.1, Sec.4.3. Assume that the strictly hyperbolic system of n equations for the vector-function $u(t, x) = (u_1, \dots, u_n)$

$$(16) \quad \partial_t u + A(u) \partial_x u = 0,$$

where $A(u)$ is a matrix with real different eigenvalues, is written in the characteristic form

$$(17) \quad l^k \left[\frac{\partial u}{\partial t} + \xi_k \frac{\partial u}{\partial x} \right] = 0,$$

where $l^k(u) = (l_1^k, \dots, l_n^k)$, $k = 1, \dots, n$, is a left eigenvector, $\xi_k(u)$ is the respective eigenvalue. If we introduce the notation $\frac{\partial u}{\partial x} = p$, $\frac{\partial u}{\partial t} = q$, we can re-write (17) as

$$(18) \quad l^k [q + \xi_k p] = 0.$$

Differentiating (18) with respect to t and x , and taking into account the condition $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial t}$ we obtain

$$(19) \quad l^k \left[\frac{\partial q}{\partial t} + \xi_k \frac{\partial q}{\partial x} \right] = \mathfrak{G}_k,$$

$$(20) \quad l^k \left[\frac{\partial p}{\partial t} + \xi_k \frac{\partial p}{\partial x} \right] = \mathfrak{F}_k,$$

where

$$\begin{aligned} \mathfrak{G}_k &= - \sum_{i,j=1}^n \left[(q_i + \xi_k p_i) \frac{\partial l_i^k}{\partial u_j} q_j + l_i^k \frac{\partial \xi^k}{\partial u_j} q_j p_i \right], \\ \mathfrak{F}_k &= - \sum_{i,j=1}^n \left[(q_i + \xi_k p_i) \frac{\partial l_i^k}{\partial u_j} p_j + l_i^k \frac{\partial \xi^k}{\partial u_j} p_j p_i \right]. \end{aligned}$$

System (17), (19), (20) is called the augmented system.

Let us denote $\mathcal{P}_k = l^k p = \sum_{i=1}^n l_i^k p_i$. Since the matrix $\Lambda = l_i^k$ is non-degenerate, we can find

$$(21) \quad p_k = \sum_{i=1}^n \lambda_i^k \mathcal{P}_i,$$

where λ_i^k are the components of Λ^{-1} . Further, from (18) we have

$$(22) \quad q_k = - \sum_{i,j=1}^n \lambda_i^k l_j^i \xi_i p_j = - \sum_{i=1}^n \lambda_i^k \xi_i \mathcal{P}_i.$$

Thus, from (20) we obtain

$$(23) \quad \frac{\partial \mathcal{P}_k}{\partial t} + \xi_k \frac{\partial \mathcal{P}_k}{\partial x} = \mathbf{F}_k,$$

where $\mathbf{F}_k = \mathfrak{F}_k + \sum_{i,j=1}^n p_i \frac{\partial l_i^k}{\partial u_j} (q_j + \xi_k p_j)$. System (23) consists of n equations, it can be considered together with system of n equations

$$(24) \quad \frac{\partial u_k}{\partial t} + \xi_k \frac{\partial u_k}{\partial x} = \mathbf{U}_k,$$

where $\mathbf{U}_k = -\sum_{i=1}^n \lambda_i^k \xi_i \mathcal{P}_i + \xi_k \sum_{i=1}^n \lambda_i^k \mathcal{P}_i$. Here p and q in \mathbf{F}_k and \mathbf{U}_k can be expressed through \mathcal{P} by formulae (21) and (22) such that

$$\mathbf{F}_k = F^k(u) + \sum_{i=1}^n F_i^k(u) \mathcal{P}_i + \sum_{i,j=1}^n F_{ij}^k(u) \mathcal{P}_i \mathcal{P}_j,$$

$$\mathbf{U}_k = U^k(u) + \sum_{i=1}^n U_i^k(u) \mathcal{P}_i, \quad i = 1, \dots, n.$$

\mathbf{F}_k and \mathbf{U}_k are quadratic and linear polynomials with respect to \mathcal{P} with coefficients that depend only on u .

Thus, the quadratically nonlinear system of $2n$ equations (23), (24) is written in the Riemann invariants.

4.2. Gas dynamics equations. First of all we re-write system (1),(2), (5) in terms of velocity v , specific volume $\tau = \frac{1}{\rho}$ and pressure p . We enumerate the components of solution as $u_1 = v$, $u_2 = \tau$, $u_3 = p$. The resulting system is

$$(25) \quad \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_x u_3 = 0,$$

$$(26) \quad \partial_t u_2 + u_1 \partial_x u_2 - u_2 \partial_x u_1 = 0,$$

$$(27) \quad \partial_t u_3 + u_1 \partial_x u_3 + \gamma u_3 \partial_x u_1 = 0.$$

The initial data $u^0(x) = (u_1^0, u_2^0, u_3^0)$ can be expressed through (6).

Thus, system (25)-(27) has the form (16), where

$$A(u) = \begin{bmatrix} u_1 & 0 & u_2 \\ -u_2 & u_1 & 0 \\ \gamma u_3 & 0 & u_1 \end{bmatrix},$$

with eigenvalues

$$\xi_1 = u_1 - \sqrt{\gamma u_3 u_2}, \quad \xi_2 = u_1, \quad \xi_3 = u_1 + \sqrt{\gamma u_3 u_2},$$

and respective left eigenvectors

$$l^1 = \begin{bmatrix} 1 & 0 & -s^{-1} \end{bmatrix}, \\ l^2 = \begin{bmatrix} 0 & s & s^{-1} \end{bmatrix},$$

$$l^3 = \begin{bmatrix} 1 & 0 & s^{-1} \end{bmatrix},$$

where $s = \sqrt{\gamma u_3/u_2}$. Thus, $\mathcal{P}_1 = p_1 - p_3/s$, $\mathcal{P}_2 = sp_2 + p_3/s$, $\mathcal{P}_3 = p_1 + p_3/s$, and after computation we obtain (23) and (24) with

$$\mathbf{F}_1 = -\frac{\gamma+1}{4}\mathcal{P}_1^2 + \frac{1}{4}\mathcal{P}_1\mathcal{P}_2 - \frac{3-\gamma}{4}\mathcal{P}_1\mathcal{P}_3 - \frac{1}{4}\mathcal{P}_2\mathcal{P}_3,$$

$$\mathbf{F}_2 = -\frac{\gamma+1}{4}\mathcal{P}_2(\mathcal{P}_1 + \mathcal{P}_3),$$

$$\mathbf{F}_3 = -\frac{\gamma+1}{4}\mathcal{P}_3^2 + \frac{1}{4}\mathcal{P}_1\mathcal{P}_2 - \frac{3-\gamma}{4}\mathcal{P}_1\mathcal{P}_3 - \frac{1}{4}\mathcal{P}_2\mathcal{P}_3,$$

$$\mathbf{U}_1 = -\sqrt{\gamma u_2 u_3}\mathcal{P}_3, \quad \mathbf{U}_2 = \frac{u_2}{2}\mathcal{P}_1 + \frac{u_2}{2\gamma u_3}\mathcal{P}_3, \quad \mathbf{U}_3 = -\gamma u_3\mathcal{P}_1,$$

subject to initial data

$$(28) \quad \mathcal{P}_k \Big|_{t=0} = \mathcal{P}_k(u^0(x)), \quad u_k \Big|_{t=0} = u_k^0(x), \quad k = 1, 2, 3,$$

We can see that the left hand sides $\mathbf{F}_k = \mathbf{F}_k(\mathcal{P})$ do not depend of u . This is the key point that allows us to prove the theorem.

Now let us consider all component of solution \mathcal{P}, u as functions of t and $\bar{x} = (x_1, x_2, x_3)$. We denote these new functions as $\bar{\mathcal{P}}, \bar{u}$ and change the system (23), (24) to

$$(29) \quad \frac{\partial \bar{\mathcal{P}}_k}{\partial t} + \sum_{i=1}^3 \xi_i(\bar{u}) \frac{\partial \bar{\mathcal{P}}_k}{\partial x_i} = \mathbf{F}_k(\bar{\mathcal{P}}),$$

$$(30) \quad \frac{\partial \bar{u}_k}{\partial t} + \sum_{i=1}^3 \xi_i(\bar{u}) \frac{\partial \bar{u}_k}{\partial x_i} = \mathbf{U}_k(\bar{\mathcal{P}}, \bar{u}).$$

Let us set the following Cauchy problem for (29), (30):

$$(31) \quad \bar{\mathcal{P}}_i(0, x_1, x_2, x_3) = \mathcal{P}_i^0(x_i), \quad \bar{u}_i(0, x_1, x_2, x_3) = u_i^0(x_i), \quad i = 1, 2, 3.$$

This allows us to study the solution to the Cauchy problem (29), (30), (31) along the rays

$$(32) \quad \frac{dx_k}{dt} = \xi_k(\bar{u}), \quad x_k(0) = x_k^0, \quad i = 1, 2, 3,$$

which are lines with coordinates $(x_1(t), x_2(t), x_3(t))$ in the 3d space. Along the rays (32) the system (29), (30) takes the form

$$\frac{d\bar{u}_k}{dt} = \mathbf{U}_k(\bar{\mathcal{P}}, \bar{u}),$$

$$(33) \quad \frac{d\bar{\mathcal{P}}_k}{dt} = \mathbf{F}_k(\bar{\mathcal{P}}), \quad k = 1, 2, 3,$$

moreover, the ODE system (33) can be solved separately subject to initial data $\bar{\mathcal{P}}_k(0, x_1^0, x_2^0, x_3^0)$.

Thus, we can find the criterion for a finite time blowup for the functions $\bar{\mathcal{P}}_k(t, x_1^0, x_2^0, x_3^0)$ in dependence on initial data.

If we succeed in finding of solution of (29), (30), (31), then we can get the solution to the Cauchy problem (23), (24), (28)

as

$$\begin{aligned} \mathcal{P}_k(t, x) &= \bar{\mathcal{P}}_k(t, x_1, x_2, x_3) \Big|_{x_k=x, x_j=x_0, j \neq k}, \\ u_k(t, x) &= \bar{u}_k(t, x_1, x_2, x_3) \Big|_{x_k=x, x_j=x_0, j \neq k}. \end{aligned}$$

Thus, if $\bar{\mathcal{P}}_k(t, x_1, x_2, x_3)$ blows up (does not blow up) along the ray starting from the point (x_0, x_0, x_0) , so does $\mathcal{P}_k(t, x)$ along every characteristic curves $\frac{dx_k}{dt} = \xi_k(u)$, $i = 1, 2, 3$, starting from the point x_0 , its projections to the planes $(x_2 = x_0, x_3 = x_0)$, $(x_1 = x_0, x_3 = x_0)$ and $(x_1 = x_0, x_2 = x_0)$, respectively.

The finite time blow up of $\mathcal{P}_k(t)$ implies the finite time gradient catastrophe of the first derivatives of solution u .

Thus, we will concentrate on the behavior of the function $\bar{\mathcal{P}}_k(t, x_1, x_2, x_3)$, which is governing by system (33). For the sake of simplicity we denote $\bar{\mathcal{P}}$ as \mathcal{P} .

First of all (33) implies

$$\frac{d(\mathcal{P}_1 - \mathcal{P}_3)}{d\mathcal{P}_2} = \frac{\mathcal{P}_1 - \mathcal{P}_3}{\mathcal{P}_2},$$

therefore

$$(34) \quad \mathcal{P}_2 = K(\mathcal{P}_1 - \mathcal{P}_3), \quad K = \text{const},$$

and (33) can be reduced to two equations:

$$(35) \quad \frac{d\mathcal{P}_1}{dt} = -\frac{(\gamma+1)-K}{4}\mathcal{P}_1^2 + \frac{(\gamma-3)-2K}{4}\mathcal{P}_1\mathcal{P}_3 + \frac{K}{4}\mathcal{P}_3^2,$$

$$(36) \quad \frac{d\mathcal{P}_3}{dt} = -\frac{(\gamma+1)-K}{4}\mathcal{P}_3^2 + \frac{(\gamma-3)-2K}{4}\mathcal{P}_1\mathcal{P}_3 + \frac{K}{4}\mathcal{P}_1^2.$$

Let us introduce new variables $R_1 = \frac{\mathcal{P}_1+\mathcal{P}_3}{2}$, $R_2 = \frac{\mathcal{P}_3-\mathcal{P}_1}{2}$. Then (35) and (36) result in

$$(37) \quad \frac{dR_1}{dt} = -R_1^2 + bR_2^2,$$

$$(38) \quad \frac{dR_2}{dt} = -\frac{\gamma+1}{2} R_1 R_2,$$

where $b = -\frac{\gamma-1}{2} + K$. System (37), (38) has the first integral

$$(39) \quad R_1^2 = C R_2^{\frac{4}{\gamma+1}} - \frac{2b}{\gamma-1} R_2^2,$$

with

$$C = \left(R_1^2(x_0) + \frac{2\gamma b}{\gamma-1} R_2^2(x_0) \right) R_2^{-\frac{4}{\gamma+1}}(x_0).$$

Thus, if $b > 0$, then the trajectories on the phase plane are closed and tend to the origin always except for the case $R_1(0) < 0$, $R_2(x_0) = 0$. If $b < 0$, then the trajectories tend to the origin only for $C \geq 0$ and $R_1(x_0) \geq 0$. If $b = 0$, system (37), (38) decompose and the solution is bounded if $R_1(x_0) \geq 0$. As follows from (34), the infinite value of the constant K corresponds to the case $Q_1(x_0) - Q_3(x_0) = 0$, $R_2(x_0) = 0$. Here system (37), (38) also can be reduced to one equation (37) and the solution is bounded if and only if $R_1(x_0) \geq 0$.

Thus, above conditions imply boundedness of derivatives of solution for all $t > 0$ and therefore the solution keep smoothness for all $t > 0$. They correspond to the cases (9), (7), (8), (10).

Otherwise, the derivatives go to infinity. Moreover, in all that cases (38) and (39) imply that $\frac{dR_2}{dt} \sim c_0 |R_2|^\sigma R_2$ for large R_2 with a positive constant c_0 and $\sigma > 0$. This means that R_2 (and R_1 as well) goes to infinity within a finite time and the solution has a gradient catastrophe.

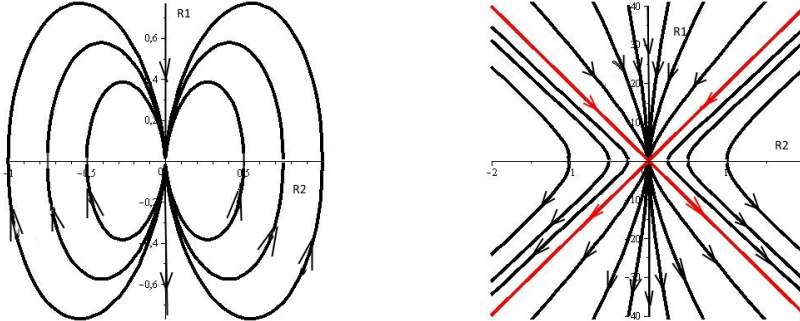


FIGURE 1. The behavior of the phase curves of system (37), (38) for $b > 0$ (left) and $b < 0$ (right).

Thus, Theorem 1 is proved.

Remark 2. Analysis of conditions (7), (8), (9), (10) shows that one can find initial density and pressure such that the solution keeps global smoothness for any initial velocity. For example, we can take $p_0 =$

$\exp x$, $\rho_0 = \exp kx$, $k = 1 + \frac{2}{\gamma}$. The data corresponds to the case (8), with $b(x) \equiv 1$, $R_2(x) \neq 0$. Such situation is not possible in the isentropic case.

Further we note that the extrema of pressure plays very important role. Indeed, in these points $R_2 = 0$ and if the derivative of velocity is negative, the points generate singularities of solution. Comparison with conditions (15) shows that for isentropic case this property also takes place.

Remark 3. Since $S = \ln \frac{p}{\rho^\gamma}$, then $K(x) = -\frac{1}{2} \frac{S'_0(x)}{\rho_0^{\gamma+1} (\ln p_0(x))'}$. Thus, (8) implies that if the entropy monotonically decrease and pressure monotonically increase (or vice versa), and $\frac{S'_0(x)}{(\ln p_0(x))'} < -(\gamma - 1) \rho_0^{\gamma+1}$, then the respective solution is globally smooth. Let us notice that the monotonic profile of entropy was used in [6] to construct smooth solutions to the compressible Euler equations.

Remark 4. The proof does not require the restriction $\gamma > 1$. In the isothermal case $\gamma = 1$ the only difference will be in the first integral (39), it should be replaced by

$$R_1^2 = CR_2^2 - 2bR_2^2 \ln |R_2|,$$

with

$$C = (R_1^2(x_0) + 2b \ln |R_2(x_0)|) R_2^{-2}(x_0).$$

The conclusion on Theorem 1 holds as well.

The result can be easily modified for any $\gamma \neq 0$. In particular, for $\gamma = -1$ (the Chaplygin gas), where the initial system is weakly nonlinear, (37), (38) imply $R_2 = R_2(x_0)$, $\frac{dR_1}{dt} = -R_1^2 + bR_2^2(x_0)$, $b = 1 + K(x_0)$. For the isentropic case ($K(x) \equiv 0$) the function R_1 goes to $-\infty$ within a finite time if

$$(40) \quad R_1(x_0) < -|R_2(x_0)|$$

for a certain $x_0 \in \mathbb{R}$. This means that the strict hyperbolicity fails on the respective solution. The singularity that arises in this case comprises a delta-like singularity in the component of density [10]. In particular, our results imply that the sufficient conditions of singularity formation from smooth initial data found in [10] are far to be exact. Condition (40) corresponds to hypothesis II from [10].

Remark 5. A natural question is whether it is possible to extend the method to the case of several space variables. The answer is positive

for the hyperbolic system of the form

$$\partial_t u + \sum_{k=1}^n A_k \partial_{x_k} u = 0, \quad u = (u_1, \dots, u_n), \quad x \in \mathbb{R}^n,$$

where the matrices A_k have a joint set of left eigenvectors. In the framework of gas dynamics this will be only in the case $p = 0$. The resulting system contains the multidimensional non-viscous Burgers equation. The criterium for the singularity formation in the smooth solution to the Cauchy problem is known for this vectorial equation [12], [1]. Therefore here the method does not give us anything new.

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